Dynamical Evolution of Rotating Stellar Systems: Fokker-Planck and N-body

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A globular cluster M15
Stellar Systems

- Star clusters (open, globular), galactic nuclei, galaxies, clusters of galaxies
- Self-gravitating
- Composed of mostly point masses
- Weak field (i.e., Newtonian dynamics)

Question: What is the fate of the self-gravitating system of point masses interacting according to Newton’s laws?
Globular Clusters Dynamics

- Oldest objects in the Galaxy: astronomical fossil
- Mostly “spherical”: ideal target for study of N-body dynamics
- Statement of problem:
  What is the fate of the self-gravitating system of point masses interacting according to Newton’s laws?
Thermodynamic properties

Moment of inertia: \( I = \eta Mr^2 \)

Temperature: \( kT = \frac{1}{2} m \sigma^2 \)

Virial theorem:
\[ E = K + V = -3NkT = -\zeta \frac{GM^2}{r} \]

Specific heat: \( \frac{\partial E}{\partial T} < 0 \)

Specific moment of inertia: \( \frac{\partial I}{\partial T} < 0 \)

Self-gravitating systems are thermodynamically unstable
Consequences

- The negative specific heat drives the “core collapse”
- The negative specific moment of inertia make the system unstable, and thus further drives the collapse of the central part.
- More detailed course of evolution can be followed only by numerical methods
General Evolution of Simple, Self-gravitating Systems

- The central density becomes infinite in finite time: gravothermal catastrophe.
- In real systems, infinite density would never happen.
- Many collisional effects become important.

Based on Fokker-Planck equation.
Methods of Dynamical Evolution

- Direct integration of equation of motion (N-body method): very time consuming, only up to $N \times 10^5$ systems, whereas we are interested in $N > 10^6$ systems.

- Statistical methods: Fokker-Planck equation, Gas dynamical approach, Monte Carlo method: Fast, but many restrictions.
  - spherical symmetry
  - velocity dispersion tensor
  - external field
Integrals of Motion and Jeans Theorem

**Integrals of motion**: a set of constants of motion that isolates the orbital properties:

- **Spherical system**: Energy (E) and angular momentum (J)
- **Spherical system with isotropic velocity**: E
- **Rotating Axisymmetric systems**: E, J along the axis of rotation ($J_z$), and $I_3$ (unknown third integral).

**Jeans Theorem**: The distribution function of collisionless systems depends on phase space variables only through the integrals of motion.
Adiabatic Invariants

- Physical quantities that remain nearly constant for slowly varying potential.
- In spherical potential, radial action $Q(E,J)$, in addition to $J$, is an adiabatic invariant.

\[ Q = 2 \int_{r_-}^{r_+} v_r \, dr \]

$f(Q,J)$ remains fixed with small changes in potential.
Fokker-Planck Equation 1.

- **Jeans Theorem:**
  \[ f = f ( I_1, I_2, ...; t ) \]

- **Effects of Stellar Encounters**
  The collisional effects are assumed to be dominated by distant encounters (accumulation of small changes in velocity).

Evolution of distribution function in time

→ **Diffusion equation in phase space**
Fokker-Planck equation 2.

Schematic form of Fokker-Planck equation in a flux conserved form

\[ 4\pi^2 p \frac{\partial f}{\partial t} = -\frac{\partial}{\partial I_1} F_{I_1} - \frac{\partial}{\partial I_2} F_{I_2} - \ldots \]

where \( F_I \) is the particle flux in integral \( I \).

\[ F_{I_i} = -D_{I_i} f - D_{I_i I_j} \frac{\partial f}{\partial I_j} \]

\( D_I \)'s are called the Fokker-Planck coefficients.
Fokker-Planck equation in self-gravitating systems

- As the system evolves, the gravitational potential also changes
  → integrals (I) may change

- The Fokker-Planck equation assumes that the potential is fixed: we need some modification of the simple Fokker-Planck equation

- Self-consistent distribution function and gravitational potential must be computed
  → utilize the adiabatic invariants
A Procedure to obtain self-consistent potential

\[ f^{\text{new}}(\phi_{\text{old}}) \rightarrow \rho(f) \rightarrow \phi \]

\[ f(Q,J) \rightarrow f(E,J) \]
Application of Fokker-Planck equation

- Fokker-Planck equation has been widely used for the study of spherical systems including the effects of
  - Stellar Evolution
  - Formation/destruction of binaries
  - Inelastic (superelastic) encounters
  - External Field (constant or time varying)

\[ \frac{\partial f}{\partial t} = -\frac{\partial}{\partial E} \left( -D_E f - D_{EE} \frac{\partial f}{\partial E} - H_E f - H_{EE} \frac{\partial f}{\partial E} \right) \]

\[ -Lf + A \]
Rotating Systems

Why do we care?

- Rotation is ubiquitous among astronomical objects (tidal effects...)
- Evidences for rotation in star clusters
- What is the consequence of the rotation? (i.e., role of gravo-gyro instability)

Difficulties

- Three isolating integrals
- Not spherically symmetric anymore
Fokker-Planck equation for rotating systems

- Integrals: E, J_z
- Third integral is unknown and ignored

\[ 4\pi^2 p \frac{\partial f}{\partial t} = -\frac{\partial}{\partial I_1} F_{I_1} - \frac{\partial}{\partial I_2} F_{I_2} - \ldots \]
Interesting Issues

- Effects of rotation on dynamical evolution
- Angular momentum redistribution/transport
- Comparison with observations
- Does Fokker-Planck equation work well for rotating systems?
Rotating Fokker-Planck Code

- The basic code was developed by Einsel & Spurzem (1999).
  (Since the third integral is unknown, it is ignored)
- The Poisson equation is solved in cylindrical coordinate with azimuthal symmetry.
- Substantially improved and collisional effects, tidal boundary conditions, etc have been added by E. Kim, H. M. Lee etc.
Initial Configuration

Rotating King Models:

Two parameter family of models
Central potential: $W_0 = \frac{\phi_0}{\sigma^2}$

Rotational parameter: $\omega_0 = \frac{3\Omega_0}{\sqrt{4\pi G n_c}}$

$$f(E, J_z) = a(e^{-\beta E} - 1) \times e^{-\beta \Omega_0 J_z}$$
## Properties of Initial Models

| $W_0$ | $\omega_0$ | $r_h/r_t$ | $T/|W|$ | $e_{\text{dyn}}$ |
|-------|------------|-----------|---------|----------------|
| 0.0   | 0.15       | 0.0       | 0.0     | 0.0            |
| 6     | 0.18       | 0.035     | 0.101   | 0.285          |
| 0.6   | 0.24       | 0.101     | 0.285   |
| 0.0   | 0.26       | 0.0       | 0.0     |
| 3     | 0.8        | 0.29      | 0.035   | 0.102          |
| 1.5   | 0.35       | 0.097     | 0.267   |
Evolution of rotating clusters

Direction toward higher rotation

Rotation makes the evolution faster

Kim et al. 2002 CCP-2006
Evaporation of Stars

The stars evaporate from cluster when \( v > v_e \), where \( v_e = -2\Phi \) is escape velocity.

The mean square of escape velocity

\[
<v_e ^2> = \frac{\int \rho(x) v_e ^2 \, d^3 x}{\int \rho(x) \, d^3 x} = -2 \frac{\int \rho(x) \Phi(x) \, d^3 x}{M} = \frac{-4W}{M}
\]

where \( W \) is potential energy, and \( M \) is the total mass.

According to Virial Theorem,

\[-W = 2K = \frac{1}{2} M <v^2>\]

and therefore,

\[
<v_e ^2> = 4 <v^2>.
\]
Evaporation Rate

Suppose the stellar system reaches velocity distribution \( f(v) \) in relaxation time \( (t_{rh}) \).

The evaporation rate per relaxation time becomes,

\[
\xi_e = -\frac{t_{rh}}{M} \frac{dM}{dt} = \frac{\int_{v_e}^{\infty} f(v)d^3v}{\int_0^{\infty} f(v)d^3v}.
\]

If \( f(v) \) is a Maxwellian, \( \xi = 0.00738 \)

For tidally bounded systems, the escape velocity is reduced:

\[
<v_e^2> = 4(1 - \lambda) <v^2>
\]

where

\[
\lambda = \frac{GM}{r_i} \left/ \frac{0.8GM}{r_h} \right. = \frac{5r_h}{4r_i}
\]
Evaporation is also accelerated

Initial evaporation is greatly enhanced by rotation

Kim et al. 2002

August 29, 2006

CCP-2006
Evolution of Rotation 1.

Shape of rotation curve does not change, but the rotation velocity decreases.
Evolution of Rotation 2.

$w_0 = 6$, $\omega_0 = 0.6$

Initial

Core-collapse

Final ($\times 10$)

initial

Final

Core collapse
Comparisons with N-body

- Fokker-Planck equation ignores the third integral: could it be justified?
- Comparisons with N-body is necessary to assure the applicability of the Fokker-Planck approach
- We have carried out N-body simulations of initially rotating clusters with same initial conditions, for single and multi-mass cases
Evolution of central density

N-body and FP results are similar. FP models are disrupted more quickly: tidal boundary condition

Kim et al. 2006
Evolution of Characteristic Radii
Velocity Profiles

Velocity distribution becomes isotropic due to two-body relaxation; the third integral would eventually disappear.
Multi-Mass Models

- Multi-mass models are more complex: the energy exchange among different mass components could accelerate the evolution
- FP calculations for multi-mass rotating clusters are carried out by Kim et al. 2004
- N-body calculations by Kim et al. 2006 confirmed most of the features of FP calculations
Example: Evolution of specific angular momentum

Angular momentum for higher mass component decreases more rapidly in the early phase.
Summary and Implications

- Direct integration of Fokker-Planck equations, ignoring the third integral have been extensively carried out with tidal cut-off, binary heating, and mass spectrum.
- Rotation tends to accelerate the evolution of star clusters due to ‘gravo-gyro’ instability.
- There exists a competition in the acceleration of the evolution between the rotation and energy exchange for multi-mass models.
- Direct N-body integrations for N~10,000 give very close results with FP: ignorance of the third integral may be justified.
- More realistic calculations are carried out/planned (i.e., central black hole, etc.).