Numerical Study on Critical Exponents of Hyperbolic Ising Lattice

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The current work focuses on critical exponents of the two-dimensional ferromagnetic Ising model defined on the hyperbolic plane. The hyperbolic plane is a simply connected infinite surface in which the Gaussian curvature possesses a constant negative curvature at arbitrary points. This non-zero curvature changes the geometric symmetry of the embedded Ising lattice model, and thus possibly induces an alteration in the universality class of the system. To clarify this point, we establish regular heptagonal and triangular Ising lattices embedded in the hyperbolic plane, and extract a series of critical exponents by means of Monte Carlo simulations and the finite-size scaling analysis.

Figure 1. A heptagonal (left) and triangular (right) lattice in terms of the Poincaré disk representation. All polygons depicted in the figures are congruent in terms of the intrinsic geometry of the hyperbolic plane.

We have considered the ferromagnetic Ising model with nearest-neighbor coupling, assuming the fixed boundary condition. In the vicinity of the transition, the spontaneous magnetization and the magnetic susceptibility of the system behave as $m(T, N) \propto N^{-\beta/\mu} m_0(x)$ and $\chi(T, N) \propto N^{\gamma/\mu} \chi_0(x)$, respectively, with the argument $x = |T - T_c| N^{1/\mu}$. Here, $\beta$, $\gamma$, and $\mu$ are the critical exponents for $m$, $\chi$, and the correlation volume $\xi_V \propto |T - T_c|^{-\mu}$, respectively.

|$\beta$| $\gamma$| $\nu$| $T_c$
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<tbody>
<tr>
<td>heptagonal</td>
<td>0.61(2)</td>
<td>2.21(4)</td>
<td>3.44(6)</td>
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<tr>
<td>triangular</td>
<td>0.73(3)</td>
<td>2.26(2)</td>
<td>3.6(4)</td>
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<tr>
<td>planar</td>
<td>$1/8$</td>
<td>$7/4$</td>
<td>2</td>
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Table 1: Comparison between the resulting exponents on the hyperbolic system with those of the planar system.

Table 1 summarizes numerical results of a series of critical exponents. Importantly, all the exponents take values different from the exact solution of the planar Ising model (The value $\mu = 2$ of the planar model is based on the relation $\mu = \nu d$ with $\nu = 1$ and $d = 2$)[1]. In addition to them, we have observed that the dynamic exponent of the hyperbolic system also assumes the value different from that the planar Ising model[2]. These findings indicate that the Ising model assigned on the hyperbolic plane belongs to a novel universality class distinct from that of the planar Ising model.

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REFERENCES